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A Fourier Theorem

and Its Application

to the Measurement

of Electromagnetic Fields

and Quantum Mechanical States

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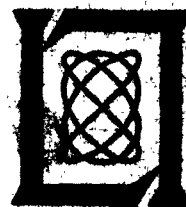LINCOLN LABORATORY

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A FOURIER THEOREM AND ITS APPLICATION TO THE MEASUREMENT OF ELECTROMAGNETIC FIELDS AND QUANTUM MECHANICAL STATES

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ABSTRACT

We shall give a mathematical theorem by which one can determine in an essentially unique way a complex function of a real argument from its absolute value and the absolute values of the Fourier transform of the truncated function for all possible truncations.

The absolute values of the function and of the Fourier transforms have a physical significance in electromagnetic and quantum theory. The theorem we present enables one to assign electric fields to energy densities and quantum mechanical states to sets of probability densities.

The measurements required for use in quantum mechanics can be expressed as the mean values of certain operators constructed from the position and momentum operators. These will be given also.

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A FOURIER THEOREM AND ITS APPLICATION TO THE MEASUREMENT OF ELECTROMAGNETIC FIELDS AND QUANTUM MECHANICAL STATES

I. INTRODUCTION

In problems of wave propagation or quantum mechanics in one dimension, one has a differential equation for a function $f(x,t)$ which is found under certain initial conditions. The function $f(x,t)$, which is generally complex, is called the wave function. For example, in the propagation of electromagnetic radiation along the x-axis, $f(x,t)$ could be chosen to be a component of the electric vector, or in quantum mechanics it could be taken as the wave function that gives the state of a one-dimensional system.

Since we shall consider one time only, we shall drop the time variables.

Although the differential equations determine $f(x)$, the measurable quantities are $|f(x)|^2$. In the case of electromagnetic wave propagation, for example, this quantity gives the energy density of the radiation, while in quantum mechanics it gives the relative probability density of finding a particle on the x-axis.

Let us define the Fourier transform $F(x)$ of $f(x)$ by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(y) e^{ixy} dy$$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-ixy} dy \quad . \tag{1}$$

The quantity $|F(x)|^2$ is also an energy density or probability density. In the electromagnetic case $|F(x)|^2$ gives the energy density in frequency space, while in quantum mechanics the quantity gives the probability density in momentum space.

It might seem that if one knew the energy density in frequency space as well as the energy density in x-space (in the electromagnetic case) or the probability densities in both coordinate and momentum space (in the quantum mechanical case), the wave function would be essentially unique. One has the feeling that all information which the wave function can provide has been used. However, a counterexample shows that the densities in the two spaces are not sufficient to characterize the wave function uniquely.[†]

Let

$$f_1(x) = \varphi_0(x) + i\varphi_2(x)$$

$$f_2(x) = \varphi_0(x) - i\varphi_2(x) \quad , \tag{2}$$

where the functions φ_n are the Hermite functions

$$\varphi_n(x) = \exp\left[-\frac{x^2}{2}\right] H_n(x) \quad ,$$

$H_n(x)$ being the Hermite polynomial of order n. Denoting the Fourier transforms of f_1 and f_2 by F_1 and F_2, respectively, we have

$$F_1(x) = \varphi_0(x) - i\varphi_2(x)$$

$$F_2(x) = \varphi_0(x) + i\varphi_2(x) \quad . \tag{3}$$

Although $|F_1(x)| = |F_2(x)|$ and $|f_1(x)| = |f_2(x)|$, $f_1(x)$ and $f_2(x)$ are linearly independent.

Hence, the information with which we are provided is, in general, insufficient to determine the function uniquely. More information is needed. This information might be provided in several forms. In Refs. 1 and 2, for example, one restricts f(x) to the class of functions that vanish identically for values of x which are sufficiently small. The additional information would then be provided by the position of the zeros in the complex plane of the Fourier transform F(x). However, we should like to restrict the additional information required to that which could be obtained from energy density or probability density measurements. In principle, these measurements can be obtained through the use of idealized existing apparatus.

II. THE PRINCIPAL THEOREM

It will be convenient to use the notion of truncated functions, i.e., functions that vanish identically when x is in an interval. Accordingly, we shall introduce the step function $\eta(x)$:

$$\eta(x) = 1 \quad , \quad x > 0$$

$$= 0 \quad , \quad x < 0 \quad . \tag{4}$$

Then

$$\eta(x - a)\, f(x) = f(x) \quad , \quad x > a$$

$$= 0 \quad , \quad x < a \tag{5a}$$

and

$$\eta(x - a)\, F(x) = F(x) \quad , \quad x > a$$

$$= 0 \quad , \quad x < a \quad . \tag{5b}$$

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Let $F_a(x)$ be the Fourier transform of $\eta(x - a)\, f(x)$:

$$F_a(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \eta(y - a)\, f(y)\, e^{ixy}\, dy = \frac{1}{\sqrt{2\pi}} \int_a^{\infty} f(y)\, e^{ixy}\, dy \quad . \tag{6}$$

Furthermore, let us define $f_a(x)$ as the Fourier transform of $\eta(x - a)\, F(x)$:

$$f_a(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \eta(y - a)\, F(y)\, e^{-ixy}\, dy = \frac{1}{\sqrt{2\pi}} \int_a^{\infty} F(y)\, e^{-ixy}\, dy \quad . \tag{7}$$

Then our principal theorem is the following.

Theorem A.

Let $f_1(x)$ and $f_2(x)$ be complex functions of the real variable $-\infty < x < \infty$, which are L^2 integrable in the Lebesgue sense. Let $F_{1a}(x)$ and $F_{2a}(x)$ be the Fourier transforms of $\eta(x - a)\, f_1(x)$ and $\eta(x - a)\, f_2(x)$, respectively. There exists a complex number ω of modulus unity such that

$$f_1(x) = \omega f_2(x) \tag{8}$$

almost everywhere if and only if almost everywhere

$$|f_1(x)| = |f_2(x)| \quad , \tag{9}$$

$$|F_{1a}(x)| = |F_{2a}(x)| \quad , \quad \text{for all } a \quad . \tag{10}$$

There is a theorem complementary to this one that is obtained by using $F(x)$ and $f_a(x)$.

Theorem B.

Let $F_1(x)$ and $F_2(x)$ be the Fourier transforms of $f_1(x)$ and $f_2(x)$, respectively [Eq. (1)]. Let $f_{1a}(x)$ and $f_{2a}(x)$ be the Fourier transforms of $\eta(x - a)\, F_1(x)$ and $\eta(x - a)\, F_2(x)$, respectively [Eq. (7)]. Then there exists a complex number ω of modulus unity such that

$$f_1(x) = \omega f_2(x) \tag{11}$$

if and only if almost everywhere

$$|F_1(x)| = |F_2(x)| \quad , \tag{12}$$

$$|f_{1a}(x)| = |f_{2a}(x)| \quad , \quad \text{for all } a \quad . \tag{13}$$

We shall prove Theorem A. Theorem B is proved analogously.

The necessary condition is obvious. Hence, we shall prove only the sufficiency. From Eq. (10)

$$\int_{-\infty}^{+\infty} F_{1a}^*(x)\, e^{i\beta x}\, F_{1a}(x)\, dx = \int_{-\infty}^{+\infty} F_{2a}^*(x)\, e^{i\beta x}\, F_{2a}(x)\, dx \quad , \quad \text{for all real } a \text{ and } \beta \quad . \tag{14}$$

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From Parseval's theorem, Eq. (11) leads to

$$\int_a^\infty f_1^*(x)\, f_1(x + \beta)\, dx = \int_a^\infty f_2^*(x)\, f_2(x + \beta)\, dx \quad , \tag{15}$$

or

$$\int_a^b f_1^*(x)\, f_1(x + \beta)\, dx = \int_a^b f_2^*(x)\, f_2(x + \beta)\, dx \quad , \tag{16}$$

where Eq. (16) holds for all real a, b and β (Theorem 64, Ref. 3).

From the lemma on page 360 of Ref. 4 it follows that

$$f_1^*(x)\, f_1(x + \beta) = f_2^*(x)\, f_2(x + \beta) \tag{17}$$

almost everywhere. Let us write f_1 and f_2 in polar form:

$$f_1(x) = A(x)\, e^{i\varphi_1(x)}$$

$$f_2(x) = A(x)\, e^{i\varphi_2(x)} \quad , \tag{18}$$

where $|f_1(x)| = |f_2(x)| = A(x)$ and where $\varphi_1(x)$ and $\varphi_2(x)$ are real.

Then for values of x and y such that $f_1(x) \neq 0$ and $f_1(y) \neq 0$,

$$e^{i[\varphi_1(x)-\varphi_1(y)]} = e^{i[\varphi_2(x)-\varphi_2(y)]} \tag{19}$$

almost everywhere.

Now, from Eq. (18)

$$f_1(x) = e^{i[\varphi_1(x)-\varphi_2(x)]}\, f_2(x) \quad . \tag{20}$$

After defining $\omega(x)$ by

$$\omega(x) = e^{i[\varphi_1(x)-\varphi_2(x)]} \quad ,$$

we have from Eq. (19)

$$\omega(x) = \omega(y) \tag{21}$$

almost everywhere for x and y such that $f_1(x) \neq 0$ and $f_1(y) \neq 0$, from which the theorem follows.

III. APPLICATION OF THE THEOREMS

We can now say how the theorems can be used to assign wave functions to probability or energy densities.

First, consider Theorem A. In the electromagnetic case $|f(x)|^2$ gives the energy density in position space. The quantity $|F_a(x)|^2$ gives the energy density in frequency space if a shutter

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has been inserted at x = a in position space so that that portion of the wave for x < a does not contribute to the energy density in frequency space. Then if f(x) is real, as it is usually taken to be in the electromagnetic case, it is determined within a sign by the energy measurements in position space and by those in frequency space for all positions of the shutter.

In the case of quantum mechanics, $|f(x)|^2$ gives the probability density in coordinate space, while $|F_a(x)|^2$ gives a priori probability density in momentum space if a particle had been found in the interval $a < x < \infty$, the state before any measurement being given by f(x). Thus, these probability densities determine f(x) within a factor ω. But since $\omega f(x)$ corresponds to the same state as f(x), these densities are in a one-to-one correspondence with states of a quantum mechanical system.

Theorem B makes use of complementary information. The energy density in frequency space is given by $|F(x)|^2$, while $|f_a(x)|^2$ is the density in position space if a filter is used to cut out frequencies less than a. Again, such densities for all a determine the electric field f(x) within a sign. Similarly, in the quantum mechanical case $|F(x)|^2$ is the probability density in momentum space. The quantity $|f_a(x)|^2$ is the density in coordinate space if it is ascertained that particles with momentum less than a have been filtered out. Such probability density measurements for all a correspond in a unique way to states.

IV. A QUANTUM MECHANICAL INTERPRETATION OF THE THEOREM

We shall give briefly a quantum mechanical interpretation of the theorem. Let us consider a quantum mechanical system in which two hypermaximal operators X and P form an irreducible set such that

$$XP - PX = iI \quad , \tag{22}$$

where I is the identity operator. The coordinate operator is taken to be X, and P is the momemtum operator. It is well known from the work of von Neumann and others that to every vector Φ in an abstract Hilbert space we can assign a pair of functions f(x) and F(x), which are Fourier transforms of each other [Eq. (1)] so that

$$\Phi \longleftrightarrow f(x) \longleftrightarrow F(x) \tag{23}$$

implies

$$e^{i\beta P} \Phi \longleftrightarrow f(x + \beta) \longleftrightarrow e^{i\beta x} F(x) \quad , \tag{24}$$

$$\eta(X - a) \Phi \longleftrightarrow \eta(x - a) f(x) \longleftrightarrow F_a(x) \quad , \tag{25}$$

$$\eta(P - a) \Phi \longleftrightarrow f_a(x) \longleftrightarrow \eta(x - a) F(x) \quad , \tag{26}$$

where F_a and f_a are defined by Eqs. (6) and (7).

It will be convenient to introduce the following definition. Two vectors (Φ_1, Φ_2) of a Hilbert space are said to belong to the same ray if a nonzero scaler ω exists such that

$$\Phi_1 = \omega \Phi_2 \quad . \tag{27}$$

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The content of the theorem can now be stated in terms of expectation values of certain operators. As usual, we define the expectation value of an operator M when a dynamical system is in a state given by the ray $\omega\phi$ where ϕ is a vector in the ray:

$$\overline{M} = (\phi, M\phi)/(\phi, \phi) \quad ,$$

using the physicist's convention in the ordering of the inner product.

The theorem then says that the states of a one-dimensional system are in a one-to-one correspondence with the set of real numbers,

$$\overline{\eta(X-a)} \quad \text{and} \quad \overline{\eta(X-b)\,\eta(P-a)\,\eta(X-b)} \quad , \quad \text{for all real a and b} \quad .$$

An alternative formulation says that the states are in a one-to-one correspondence with the set of complex numbers $e^{i\alpha P}\,\eta(X)\,e^{i\beta P}$ for all real α and β.

ACKNOWLEDGEMENTS

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